

Univalence in locally cartesian closed ∞ -categories

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Abstract

In the setting of presentable locally cartesian closed ∞ -categories, we show that univalent families, in the sense of Voevodsky, form a poset isomorphic to the poset of bounded local classes, in the sense of Lurie. It follows that every ∞ -topos has a hierarchy of “universal” univalent families, indexed by regular cardinals, and that n -topoi have univalent families classifying $(n - 2)$ -truncated maps. We show that univalent families are preserved (and detected) by right adjoints to locally cartesian localizations, and use this to exhibit certain canonical univalent families in ∞ -quasitopoi (∞ -categories of “separated presheaves”). We also exhibit some more exotic examples of univalent families, illustrating that a univalent family in an n -topos need not be $(n - 2)$ -truncated, as well as some univalent families in the Morel-Voevodsky ∞ -category of motivic spaces, an instance of a locally cartesian closed ∞ -category which is not an n -topos for any $0 \leq n \leq \infty$. Lastly, we show that any presentable locally cartesian closed ∞ -category is modeled by a type-theoretic model category, and conversely that the ∞ -category underlying a type-theoretic model category is presentable and locally cartesian closed; moreover, univalent families in presentable locally cartesian closed ∞ -categories correspond to univalent fibrations in type-theoretic model categories.

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Introduction

The connection between type theory and homotopy theory, which goes back to Hofmann and Streicher [11], with more recent advances having been made notably by Awodey and Warren [3], Gambino and Garner [8], van den Berg and Garner [18], and Lumsdaine [14], has reached a new level of significance with Voevodsky’s Univalence Axiom [19], [13], which roughly stipulates that intensional identity is homotopy equivalence. This may potentially provide a new foundation for mathematics in which the notion of homotopy is built in at the foundational level, reflecting the important mathematical practice of identifying algebraic structures if they are isomorphic, categories if they are equivalent, and so on. On the categorical level, the existence of a univalent universe also solves the so-called coherence problem in categorical semantics of type theory: a coherent choice of all type-theoretic operations can be made in terms of the universe and exploiting its universal property. Voevodsky explained his Univalent Foundations Program in a series of 9 lectures in Oberwolfach in 2011 [2], and established in particular a categorical model containing a univalent universe: it is a certain Kan fibration in the category of simplicial sets, and its univalence property is established

using Quillen model structures and a well-ordering trick. A related construction is due to Streicher, and some simplifications in Voevodsky’s proof were provided by Joyal and by Moerdijk. A concise and self-contained exposition of the proof was recently given by Kapulkin, Lumsdaine and Voevodsky [13]. Recently Shulman [16], building on Voevodsky’s result, has shown that categories of inverse diagrams of simplicial sets provide other models.

The difficulty encountered in constructing these univalent universes stems from the fact that many features of type theory are not invariant under homotopy equivalence, which makes constructions delicate. But in fact, the univalence axiom itself is a very robust notion (and homotopy invariant, in particular).

In the present contribution we study univalence for its own sake from the intrinsic viewpoint of presentable locally cartesian closed ∞ -categories. We prove some general results about univalent families, and provide, rather easily, a rich supply of models for the Univalence Axiom, including univalent families in all ∞ -topoi. Only once these univalent families are constructed at the level of ∞ -categories do we consider the issue of lifting them, by a fairly standard procedure (see Section 6), to the model categories where current semantics of type theory takes place. We hasten to point out that the univalent fibrations produced are not necessarily universes, though, and hence do not immediately provide new models for type theory in the way Voevodsky’s and Shulman’s constructions do. It is expected that every ∞ -topos should be a model for type theory, and we hope that the present groundwork on the level of ∞ -categories will prove useful to establish the strict models. More generally we speculate that eventually a genuinely ∞ -categorical semantics for type theory will be developed, bypassing altogether the subtleties inherent in model categorical semantics.

We briefly outline our results. We establish in Corollary 2.9 that univalent families form a poset isomorphic to the poset of bounded local classes of maps in the sense of Lurie.¹ It follows in particular that in every ∞ -topos and for each regular cardinal κ , the universal family classifying relatively κ -compact maps is a univalent family. The relationship between univalence and the ∞ -topos axioms was first pointed out by Joyal [2]. The Univalence Axiom can be interpreted as a descent property which allows to glue together families into “moduli spaces”. Having this descent property for *all* families characterizes precisely the ∞ -topoi among presentable locally cartesian closed ∞ -categories: in an ∞ -topos the class of all maps is local, and each regular cardinal yields a univalent family, a “universe”. Similarly, in any n -topos there is a hierarchy of univalent families classifying (bounded) $(n - 2)$ -truncated maps.

It is likely that the case of ∞ -topoi will be the main case of interest to type theory. Nevertheless, as the notion of univalence is fundamental, it is interesting to investigate it for its own sake, and to provide further examples of univalent families. The natural setting for this study is that of presentable locally cartesian closed ∞ -categories, not only because of the obvious importance in type theory of this class of ∞ -categories, but as much because the abstract notion of univalence behaves well in this context: a fundamental result (Theorem 2.10) states that univalent families are preserved and detected by right adjoints to locally cartesian localizations. Since any presentable locally cartesian closed ∞ -category embeds in this way into an ∞ -topos, this gives some control over univalent families in general, and shows in particular that every univalent family must be a subfamily of some object classifier of the ambient ∞ -topos.

Factorization systems play an important role in our treatment. We show in Section 3 that the right class of a basechange-stable factorization system in an ∞ -topos is a local class. This leads to an easy proof of the fact that k -truncated maps form a local class, and is also a key ingredient in the treatment of ∞ -quasitopoi: in Section 4 we demonstrate that significant univalent families can exist outside the realm of topoi, exhibiting big univalent families in ∞ -quasitopoi, by which we mean ∞ -categories of \mathcal{F} -separated objects for a suitable factorization system $(\mathcal{E}, \mathcal{F})$, and with respect to a locally cartesian localization.

While the constructions so far are “top-down”, it is also interesting to construct univalent families “bottom-up”, which is the topic of Section 5. The smallest univalent families are bundle classifiers: for every object F of an ∞ -topos, the universal bundle with fiber F is univalent (see 5.2). Bigger families can be obtained by taking unions. This viewpoint leads to some unexpected univalent families, exemplifying in particular that a univalent family in an n -topos need not be $(n - 2)$ -truncated (see 5.5). We also exhibit some univalent families in the locally cartesian closed ∞ -category of motivic spaces [15] constructed from certain group schemes (see 5.10 and 5.11), providing further examples of univalent families outside of topoi.

¹[HTT, Section 6.2]

The current categorical semantics of type theory involves certain strict fibration properties, which are necessary to get a literal interpretation of the syntactic rules. These strictness features have no intrinsic homotopical content, but can be formulated in terms of 1-categorical notions in a Quillen model category. We observe in the final Section 6 that one can always model univalent families in presentable locally cartesian closed ∞ -categories by univalent fibrations (although not necessarily universes in the sense of type theory, cf. [16]) in what we call type-theoretic model categories, namely right-proper combinatorial Quillen model categories whose underlying category is locally cartesian closed, and whose cofibrations are precisely the monomorphisms.

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Terminology and notation. For simplicity we adhere to the terminology and notation of Lurie [HTT] as much as possible.

1 Locally cartesian closed ∞ -categories

1.1 Presentable locally cartesian closed ∞ -categories. Recall that an ∞ -category \mathcal{C} is *locally cartesian closed* when for any $f : T \rightarrow S$ in \mathcal{C} , the pullback functor $f^* : \mathcal{C}_{/S} \rightarrow \mathcal{C}_{/T}$ has a right adjoint. We will often assume that \mathcal{C} is presentable; in this case, by the adjoint functor theorem² (since slices of presentable ∞ -categories are again presentable), being locally cartesian closed is equivalent to colimits being universal, which is the condition usually used in topos theory.

1.2 Locally cartesian localizations. Just as left-exact localizations are a central notion in topos theory, the notion of locally cartesian localization,³ introduced here, plays a similar role for locally cartesian closed ∞ -categories. Let \mathcal{P} be a presentable locally cartesian closed ∞ -category and let $L : \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$ be an accessible localization, with right adjoint inclusion functor $G : \mathcal{C} \rightarrow \mathcal{P}$. We refer to the objects of \mathcal{C} as local objects. For each local object S there is induced a localization functor $L_S : \mathcal{P}_{/S} \rightarrow \mathcal{C}_{/S} \subseteq \mathcal{P}_{/S}$ (with right adjoint inclusion functor G_S). For any map $f : T \rightarrow S$ between local objects we have commutative diagrams

$$\begin{array}{ccc} \mathcal{P}_{/S} & \xrightarrow{L_S} & \mathcal{C}_{/S} \\ f_! \uparrow & & \uparrow f_! \\ \mathcal{P}_{/T} & \xrightarrow{L_T} & \mathcal{C}_{/T} \end{array} \quad \begin{array}{ccc} \mathcal{P}_{/S} & \xleftarrow{G_S} & \mathcal{C}_{/S} \\ f^* \downarrow & & \downarrow f^* \\ \mathcal{P}_{/T} & \xleftarrow{G_T} & \mathcal{C}_{/T}. \end{array}$$

We say that the localization L is *locally cartesian* if it commutes with basechange between local objects. In other words, for all $f : T \rightarrow S$ in \mathcal{C} , the diagram

$$\begin{array}{ccc} \mathcal{P}_{/S} & \xrightarrow{L_S} & \mathcal{C}_{/S} \\ f^* \downarrow & & \downarrow f^* \\ \mathcal{P}_{/T} & \xrightarrow{L_T} & \mathcal{C}_{/T} \end{array}$$

commutes. Equivalently, for every diagram $T \rightarrow S \leftarrow X$ in \mathcal{P} (where S and T are local objects), the natural map

$$L(T \times_S X) \longrightarrow LT \times_{L_S} LX \simeq T \times_S LX$$

is an equivalence.

²[HTT 5.5.2.9]

³Called *semi-left-exact localization* by Cassidy–Hébert–Kelly [5] for 1-categories.

Proposition 1.3 *Let \mathcal{P} be a presentable locally cartesian closed ∞ -category and let $L : \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$ be an accessible localization. The following conditions are equivalent:*

1. *L is locally cartesian*
2. *For every morphism $f : T \rightarrow S$ between local objects, the functor $f^* : \mathcal{P}_S \rightarrow \mathcal{P}_T$ preserves local equivalences. That is, given a pullback diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\quad} & X \\ f^*p \downarrow & \lrcorner & \downarrow p \\ T & \xrightarrow{\quad f \quad} & S \end{array}$$

*if $L(p)$ is an equivalence, then also $L(f^*p)$ is an equivalence.*

3. *For every morphism $f : T \rightarrow S$ between local objects, the functor $f_* : \mathcal{P}_T \rightarrow \mathcal{P}_S$ sends L_T -local objects to L_S -local objects.*

Proof. (1) \Rightarrow (2): Suppose L preserves basechange between local objects. If p is a local equivalence then $L(p)$ and hence $f^*(Lp)$ are equivalences. But the latter is equivalent to $L(f^*(p))$ by assumption, which is to say that $f^*(p)$ is a local equivalence. Hence f^* preserves local equivalences.

(2) \Rightarrow (3): Assuming that f^* preserves local equivalences, let $z : Z \rightarrow T$ be a local object in \mathcal{P}_T . We need to check that $f_*(z)$ is again a local object, now in \mathcal{P}_S . So pick a local equivalence $\alpha : p \rightarrow p'$ in \mathcal{P}_S , i.e. a triangle diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & X' \\ & \searrow p & \swarrow p' \\ & S. & \end{array}$$

We have a diagram

$$\begin{array}{ccc} \mathrm{Map}_T(f^*(p'), z) & \xrightarrow{\quad} & \mathrm{Map}_T(f^*(p), z) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathrm{Map}_S(p', f_*(z)) & \xrightarrow{\quad} & \mathrm{Map}_S(p, f_*(z)) \end{array}$$

where the vertical maps are equivalences by adjunction. Since f^* preserves local equivalences, the top horizontal map is an equivalence, and hence the bottom horizontal map is an equivalence. Since this is true for all local equivalences $\alpha : p \rightarrow p'$, this is to say that $f_*(z)$ is a local object. Hence f_* preserves local objects.

(3) \Rightarrow (1): To say that each f_* preserves local objects means that the square of right adjoints

$$\begin{array}{ccc} \mathcal{P}_S & \xleftarrow{G_S} & \mathcal{C}_S \\ f_* \uparrow & & \uparrow f_* \\ \mathcal{P}_T & \xleftarrow{G_T} & \mathcal{C}_T \end{array}$$

commutes. The analogous square of left adjoints then also commutes, which is precisely what it means for L to be locally cartesian. \square

Proposition 1.4 *Let \mathcal{P} be a presentable locally cartesian closed ∞ -category, and let $L : \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$ be an accessible locally cartesian localization. Then \mathcal{C} is a presentable locally cartesian closed ∞ -category.*

Proof. This follows by inspection of the proof of [HTT, Lemma 6.1.3.15]: there it is proved that a left-exact localization preserves the property of being locally cartesian closed, but in fact the only place left-exactness is used is to preserve a certain pullback square where in fact one of the legs is between local objects. \square

Recall that, for any presentable ∞ -category \mathcal{C} and any sufficiently large regular cardinal κ , the inclusion $\mathcal{C}^\kappa \rightarrow \mathcal{C}$ induces a colimit preserving functor

$$\mathrm{Pre}(\mathcal{C}^\kappa) \longrightarrow \mathcal{C}$$

which is essentially surjective and admits a fully faithful right adjoint.⁴ We refer to this as the “standard presentation” (even though it involves a choice of cardinal).

Proposition 1.5 *If \mathcal{C} is a presentable locally cartesian closed ∞ -category then its standard presentation $L : \mathrm{Pre}(\mathcal{C}^\kappa) \rightarrow \mathcal{C}$ is an accessible locally cartesian localization.*

Proof. Choose κ such that there exists a localization functor

$$L : \mathrm{Pre}(\mathcal{C}^\kappa) \longrightarrow \mathcal{C} \subseteq \mathrm{Pre}(\mathcal{C}^\kappa);$$

we must show that L is locally cartesian. We shall apply criterion (2) of Proposition 1.3. Let $f : T \rightarrow S$ be a map of local objects and let $\kappa' \geq \kappa$ be a cardinal such that both S and T are in $\mathcal{C}^{\kappa'}$. By the commutativity of the diagram

$$\begin{array}{ccccc} \mathcal{C}_{/T} & \longrightarrow & \mathrm{Pre}(\mathcal{C}^\kappa)_{/T} & \longrightarrow & \mathrm{Pre}(\mathcal{C}^{\kappa'})_{/T} \\ & & \downarrow f_* & & \downarrow f_* \\ \mathcal{C}_{/S} & \longrightarrow & \mathrm{Pre}(\mathcal{C}^\kappa)_{/S} & \longrightarrow & \mathrm{Pre}(\mathcal{C}^{\kappa'})_{/S}, \end{array}$$

we may suppose without loss of generality that $\kappa' = \kappa$ and that f is a map of representables. We need to show that f^* applied to any generating local equivalence

$$\mathrm{colim}_i \mathrm{Map}(-, x_i) \longrightarrow \mathrm{Map}(-, \mathrm{colim}_i x_i)$$

is again a local equivalence. But this is clear: writing $T \simeq \mathrm{Map}(-, t)$ and $S \simeq \mathrm{Map}(-, s)$ for some $s, t \in \mathcal{C}^\kappa$,

$$\begin{aligned} f^* \mathrm{colim}_i \mathrm{Map}(-, x_i) &\simeq \mathrm{colim}_i f^* \mathrm{Map}(-, x_i) \simeq \mathrm{colim}_i \mathrm{Map}(-, f^* x_i) \longrightarrow \\ &\mathrm{Map}(-, \mathrm{colim}_i f^* x_i) \simeq \mathrm{Map}(-, f^* \mathrm{colim}_i x_i) \simeq f^* \mathrm{Map}(-, \mathrm{colim}_i x_i) \end{aligned}$$

is itself a generating local equivalence because \mathcal{C} is locally cartesian closed, so both $f^* : \mathcal{C}_{/S}^\kappa \rightarrow \mathcal{C}_{/T}^\kappa$ and $f^* : \mathrm{Pre}(\mathcal{C}^\kappa)_{/S} \rightarrow \mathrm{Pre}(\mathcal{C}^\kappa)_{/T}$ preserve colimits. Finally, L is accessible as it is a left adjoint functor between accessible ∞ -categories.⁵ \square

Corollary 1.6 *An ∞ -category \mathcal{C} is presentable and locally cartesian closed if and only if there exists a small ∞ -category \mathcal{D} such that \mathcal{C} is an accessible locally cartesian localization of $\mathrm{Pre}(\mathcal{D})$.*

Lemma 1.7 *Let \mathcal{X} be a locally cartesian closed ∞ -category, and let $L : \mathcal{X} \rightarrow \mathcal{C} \subseteq \mathcal{X}$ be a locally cartesian localization, with right adjoint inclusion functor $G : \mathcal{C} \rightarrow \mathcal{X}$ and unit $\eta : \mathrm{id} \Rightarrow GL$. Then for each object $T \in \mathcal{X}$, the pullback functor $\eta_T^* \circ G_{LT} : \mathcal{C}_{/LT} \rightarrow \mathcal{X}_{/GLT} \rightarrow \mathcal{X}_{/T}$ is fully faithful and right adjoint to $\eta_T! \circ L_{LT} : \mathcal{X}_{/T} \rightarrow \mathcal{X}_{/GLT} \rightarrow \mathcal{C}_{/LT}$.*

⁴[HTT, Theorem 5.5.1.1]

⁵[HTT, Proposition 5.4.7.7]

Proof. It is clear that $\eta_T^* \circ G_{LT}$ is a right adjoint, as it is the composite of two right adjoints. To establish that η_T^\sharp is fully faithful, we check that for every $Z \rightarrow LT$ in $\mathcal{C}_{/LT}$, the corresponding component of the counit $\varepsilon_Z : L\eta_T^* G_{LT} Z \rightarrow Z$ is an equivalence. The Z -component of the counit $\eta_T! \circ \eta_T^* \Rightarrow \text{id}$ is the projection $T \times_{GLT} Z \rightarrow Z$. It is not in general an equivalence, so η_T^* itself is not fully faithful. But when restricted to $\mathcal{C}_{/LT}$, we do obtain a fully faithful right adjoint: indeed, precomposing with G_{LT} amounts to assuming that Z is local, and when applying L we get

$$L\eta_T^* G_{LT} Z \simeq L(T \times_{GLT} Z) \simeq LT \times_{LT} LZ \longrightarrow Z,$$

which is clearly an equivalence. \square

Lemma 1.8 *For \mathcal{C} a presentable locally cartesian closed ∞ -category, the truncation functor $L : \mathcal{C} \rightarrow \tau_{\leq k} \mathcal{C}$ is accessible and locally cartesian.*

Proof. Accessibility of L follows from [HTT 5.5.6.18]. To establish that L is locally cartesian, we need to check, for k -truncated objects S and T , that the diagram

$$\begin{array}{ccc} \mathcal{C}_{/S} & \xrightarrow{L_S} & (\tau_{\leq k} \mathcal{C})_{/S} \\ f^* \downarrow & & \downarrow f^* \\ \mathcal{C}_{/T} & \xrightarrow{L_T} & (\tau_{\leq k} \mathcal{C})_{/T} \end{array}$$

commutes. But we have natural equivalences $(\tau_{\leq k} \mathcal{C})_{/S} \simeq \tau_{\leq k}(\mathcal{C}_{/S})$ (e.g. by [HTT 5.5.6.14]), and the equivalent diagram

$$\begin{array}{ccc} \mathcal{C}_{/S} & \xrightarrow{\tau_{\leq k}} & \tau_{\leq k}(\mathcal{C}_{/S}) \\ f^* \downarrow & & \downarrow f^* \\ \mathcal{C}_{/T} & \xrightarrow{\tau_{\leq k}} & \tau_{\leq k}(\mathcal{C}_{/T}), \end{array}$$

commutes by [HTT 5.5.6.28] since f^* preserves colimits and finite limits, \mathcal{C} being locally cartesian closed. \square

1.9 Monomorphisms. Recall that a map $f : X \rightarrow Y$ in an ∞ -category is said to be a *monomorphism* if it is (-1) -truncated (i.e. its fibers are (-1) -truncated objects). This is equivalent to the condition that the diagonal $\delta_f : X \rightarrow X \times_Y X$ is an equivalence. It is also easy to see that $f : X \rightarrow Y$ is mono if and only if the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \delta_X \downarrow & & \downarrow \delta_Y \\ X \times X & \xrightarrow{f \times f} & Y \times Y \end{array}$$

is a pullback.

2 Local classes of maps and univalent families

2.1 Local classes of maps and universal families. Let \mathcal{C} be a presentable locally cartesian closed ∞ -category. Adopting the notation of Lurie [HTT, Section 6.1], we write $\mathcal{O}_{\mathcal{C}} = \text{Fun}(\Delta^1, \mathcal{C})$ for the ∞ -category of arrows in \mathcal{C} , always considered together with its codomain fibration $p : \mathcal{O}_{\mathcal{C}} \rightarrow \mathcal{C}$ (evaluation at $\{1\} \subset \Delta^1$). Its fiber over an object T is the slice ∞ -category $\mathcal{C}_{/T}$.

For a class of maps \mathcal{F} in \mathcal{C} , assumed to be stable under pullback, we denote by $\mathcal{O}_{\mathcal{C}}^{\mathcal{F}}$ the full subcategory of $\mathcal{O}_{\mathcal{C}}$ consisting of those maps which are in \mathcal{F} , and we denote by $\mathcal{O}_{\mathcal{C}}^{(\mathcal{F})}$ the category with the same objects

as $\mathcal{O}_{\mathcal{C}}^{\mathcal{F}}$ but containing only the p -cartesian arrows (the pullback squares). The codomain fibration $\mathcal{O}_{\mathcal{C}}^{(\mathcal{F})} \rightarrow \mathcal{C}$ is a right fibration, and its fiber over an object T is the ∞ -groupoid $\mathcal{C}_{/T}^{\text{eq}}$, the maximal subgroupoid of $\mathcal{C}_{/T}$. When \mathcal{F} is the class of all maps in \mathcal{C} we also write $\mathcal{O}_{\mathcal{C}}^{(\text{all})}$ for the subcategory of all objects but only the p -cartesian arrows. Straightening⁶ the right fibration $\mathcal{O}_{\mathcal{C}}^{(\mathcal{F})} \rightarrow \mathcal{C}$ determines a sheaf

$$F : \mathcal{C}^{\text{op}} \rightarrow \widehat{\text{Gpd}}_{\infty}$$

with values in the ∞ -category $\widehat{\text{Gpd}}_{\infty}$ of (possibly large) ∞ -groupoids. The class \mathcal{F} is called *local*⁷ when it is closed under basechange and F preserves small limits.

If \mathcal{F} is a local class, then the sheaf $F : \mathcal{C}^{\text{op}} \rightarrow \widehat{\text{Gpd}}_{\infty}$ is representable if and only if it takes small values.⁸ This can be ensured by a cardinal bound: for a regular cardinal κ , let \mathcal{F}_{κ} denote the class of relatively κ -compact maps in \mathcal{F} . If κ is sufficiently large then \mathcal{F}_{κ} is again a local class,⁹ called a bounded local class. In this case the corresponding sheaf F_{κ} is representable,¹⁰ and the representing object $S_{\mathcal{F}_{\kappa}}$ is called a classifying object for the class. It carries a *universal family*, i.e. a map $X_{\mathcal{F}_{\kappa}} \rightarrow S_{\mathcal{F}_{\kappa}}$ which is terminal in $\mathcal{O}_{\mathcal{C}}^{(\mathcal{F}_{\kappa})}$.

Recall that $r : \mathcal{C}_{/S} \rightarrow \mathcal{C}$ is the right fibration whose fiber over T is $\text{Map}(T, S)$ and whose associated sheaf $\mathcal{C}^{\text{op}} \rightarrow \widehat{\text{Gpd}}_{\infty}$ is represented by S . Also recall that, by the Yoneda lemma, a map $p : X \rightarrow S$ determines a map of sheaves

$$\mathcal{C}_{/S} \rightarrow \mathcal{O}_{\mathcal{C}}^{(\text{all})}.$$

The image of a map $f : T \rightarrow S$ is a pullback $f^*X \rightarrow T$. Similarly, if $p : X \rightarrow S$ belongs to a local class \mathcal{F} , then the projection $(\mathcal{O}_{\mathcal{C}}^{(\mathcal{F})})_{/p} \rightarrow \mathcal{C}_{/S}$ is a trivial Kan fibration (by the universal property of the pullback); choosing a section (which amounts to choosing pullbacks of p), and composing with the projection $(\mathcal{O}_{\mathcal{C}}^{(\mathcal{F})})_{/p} \rightarrow \mathcal{O}_{\mathcal{C}}^{(\mathcal{F})}$ gives a map of sheaves

$$\mathcal{C}_{/S} \rightarrow \mathcal{O}_{\mathcal{C}}^{(\mathcal{F})}$$

which sends an object $f : T \rightarrow S$ to $f^*(p)$. The fiber over T of this map is $\text{Map}(T, S) \rightarrow (\mathcal{C}_{/T})^{(\mathcal{F})}$. Since the projection $(\mathcal{O}_{\mathcal{C}}^{(\mathcal{F})})_{/p} \rightarrow \mathcal{O}_{\mathcal{C}}^{(\mathcal{F})}$ is an equivalence if and only if p is terminal in $\mathcal{O}_{\mathcal{C}}^{(\mathcal{F})}$, we can conclude:

Lemma 2.2 *Suppose a map $p : X \rightarrow S$ belongs to a local class \mathcal{F} . Then the functor $\mathcal{C}_{/S} \rightarrow \mathcal{O}_{\mathcal{C}}^{(\mathcal{F})}$ which sends f to $f^*(p)$ is an equivalence if and only if p is a universal family for \mathcal{F} , i.e. is a terminal object in $\mathcal{O}_{\mathcal{C}}^{(\mathcal{F})}$. Moreover, in this case, \mathcal{F} is a bounded local class.* \square

2.3 Examples. The class of all equivalences is always local. In a presentable locally cartesian closed ∞ -category \mathcal{C} , the class of all maps is local if and only if \mathcal{C} is an ∞ -topos.¹¹ In this case there is for each sufficiently large regular cardinal κ a universal family classifying relatively κ -compact maps. These deserve to be called universes in \mathcal{C} . The class \mathcal{T}_{n-2} of $(n-2)$ -truncated maps is local in any ∞ -topos (Corollary 3.3), and also in any n -topos ([HTT 6.2.1.5]; see also Corollaries 3.8 and 3.9).

An important example of this situation is when \mathcal{C} is the 2-topos of 1-groupoids. There is then a universal family of 0-groupoids, i.e. sets. This is a suitable setting for combinatorics, and is also exemplified by classical algebraic geometry where groupoid-valued sheaves are needed to provide universal families of schemes.

2.4 Univalence. Given a map $p : X \rightarrow S$ we write

$$\underline{\text{Eq}}_{/S}(X, X) \rightarrow \mathcal{C}_{/S \times S}$$

⁶[HTT, Chapter 2]

⁷[HTT 6.1.3.8]

⁸[HTT 6.1.6.3]

⁹What can go wrong for smaller cardinals is that the class might not be closed under pullback. See [HTT 6.1.6.7] for details.

¹⁰[HTT 5.5.2.2]

¹¹Combine [HTT, Theorem 6.1.0.6] with [HTT, Theorem 6.1.3.9]

for the sheaf whose T -points $(f, g) : T \rightarrow S \times S$ form the space $\text{Eq}_{/T}(f^*X, g^*X)$ of equivalences $f^*X \simeq g^*X$ over T . We say that $p : X \rightarrow S$ is a *univalent family*¹² if the diagonal map $\delta : S \rightarrow S \times S$ represents the sheaf $\underline{\text{Eq}}_{/S}(X, X)$. That is, the natural map

$$u : \delta \longrightarrow \underline{\text{Eq}}_{/S}(X, X)$$

is an equivalence over $\mathcal{C}_{/S \times S}$.

We say that a map $q : Y \rightarrow T$ is *classified by* $p : X \rightarrow S$ if there exists $f : T \rightarrow S$ such that $f^*p = q$. Let \mathcal{F}_p denote the class of those objects of $\mathcal{O}_C^{(\text{all})}$ which lie in the essential image of the projection $(\mathcal{O}_C^{(\text{all})})_{/p} \rightarrow \mathcal{O}_C$.

Proposition 2.5 *Let $p : X \rightarrow S$ be a map in a presentable locally cartesian closed ∞ -category \mathcal{C} . Then the following conditions are equivalent.*

1. *p is univalent.*
2. *The morphism of sheaves $\mathcal{C}_{/S} \rightarrow \mathcal{O}_C^{(\text{all})}$ is a monomorphism; that is, for every object $T \in \mathcal{C}$, the map*

$$\text{Map}_{\mathcal{C}}(T, S) \longrightarrow \mathcal{C}_{/T}^{\text{eq}}$$

which sends f to $f^(p)$ is a monomorphism.*

3. *The class \mathcal{F}_p (of pullbacks of $p : X \rightarrow S$) is a bounded local class.*

Proof. We first prove $(1) \Leftrightarrow (2)$. Suppose given $(f, g) : T \rightarrow S \times S$ and consider the commutative diagram

$$\begin{array}{ccc} \text{Map}_{/S \times S}((f, g), \delta) & \longrightarrow & \text{Eq}_{/T}(f^*X, g^*X) \\ \downarrow & & \downarrow \\ \text{Map}(T, S) & \longrightarrow & \mathcal{C}_{/T}^{\text{eq}} \\ \downarrow & & \downarrow \\ \text{Map}(T, S \times S) & \longrightarrow & \mathcal{C}_{/T}^{\text{eq}} \times \mathcal{C}_{/T}^{\text{eq}} \end{array}$$

in which the vertical maps are fiber sequences (over the point (f, g)); the lower vertical maps are diagonals. The middle and bottom horizontal maps are given by pulling back $p : X \rightarrow S$. Condition (2) is equivalent to the bottom square being a pullback for all T , cf. 1.9. On the other hand, the top horizontal map is the (f, g) -fiber of $u : \delta \rightarrow \underline{\text{Eq}}_{/S}(X, X)$, so univalence amounts to saying that the top horizontal map is an equivalence for all T , and all $f, g : T \rightarrow S$. The assertion follows since a square is a pullback if and only if all fibers are equivalent.

Now for $(2) \Leftrightarrow (3)$. Suppose $\text{Map}(T, S) \rightarrow \mathcal{C}_{/T}^{\text{eq}}$ is a monomorphism and let \mathcal{F}_p denote the class of maps obtained as pullbacks of p . It is clear that the essential image of $\text{Map}(T, S) \rightarrow \mathcal{C}_{/T}^{\text{eq}}$ is precisely $\mathcal{C}_{/T}^{(\mathcal{F}_p)}$, so we have an equivalence $\text{Map}(T, S) \xrightarrow{\sim} \mathcal{C}_{/T}^{(\mathcal{F}_p)}$. Using Lemma 2.2, this means that \mathcal{F}_p is a bounded local class, as it is bounded by any regular cardinal κ bigger than all the fibers of p . Conversely, if \mathcal{F}_p is a bounded local class then it has a universal family which has to be p itself, and we have a monomorphism $\text{Map}(T, S) \xrightarrow{\sim} \mathcal{C}_{/T}^{(\mathcal{F}_p)} \rightarrow \mathcal{C}_{/T}^{\text{eq}}$. \square

Corollary 2.6 *There is a one-to-one correspondence between (equivalence classes of) univalent families, and bounded local classes of maps.*

¹²See [13], Definition 27.

Proof. Given a bounded local class of maps, the associated universal family p yields an equivalence

$$\mathrm{Map}(T, S) \longrightarrow \mathcal{C}_{/T}^{(\mathcal{F})},$$

and postcomposing with the monomorphism $\mathcal{C}_{/T}^{(\mathcal{F})} \rightarrow \mathcal{C}_{/T}^{\mathrm{eq}}$ yields a monomorphism, so p is univalent. It is clear that \mathcal{F} is precisely the class of all pullbacks of p . Conversely, given a univalent family p , the class \mathcal{F}_p of all pullbacks of p is a bounded local class, by the previous proposition, and it is clear that the universal family associated to \mathcal{F}_p is equivalent to p . \square

Theorem 2.7 *Let $p' : X' \rightarrow S'$ be the pullback of a univalent family $p : X \rightarrow S$ along a map $m : S' \rightarrow S$. Then p' is univalent if and only if m is a monomorphism.*

Proof. We use criterion (2) of Proposition 2.5: in the commutative diagram

$$\begin{array}{ccc} \mathrm{Map}(T, S') & \longrightarrow & \mathcal{C}_{/T}^{\mathrm{eq}} \\ m_! \downarrow & & \downarrow = \\ \mathrm{Map}(T, S) & \longrightarrow & \mathcal{C}_{/T}^{\mathrm{eq}} \end{array}$$

the bottom map is mono by Proposition 2.5. Hence the top map is mono if and only if $m_!$ is mono, which is to say that m is mono. \square

Corollary 2.8 *If $p' : X' \rightarrow S'$ belongs to some bounded local class in a presentable locally cartesian closed ∞ -category, then it is univalent if and only if it is the pullback of the associated universal family along a monomorphism.* \square

Note that this univalence criterion is independent of which bounded local class p' belongs to. Note also that if p' belongs to a local class \mathcal{F} , then it also belongs to the bounded local class \mathcal{F}_κ , for κ sufficiently large, and larger than any fiber of p' . Finally note that if \mathcal{C} is an ∞ -topos, every p' belongs to the local class of all maps, so we get a complete classification of univalent families in this case.

It follows from 2.7 that (equivalence classes of) univalent families in \mathcal{C} form a (possibly large) poset. On the other hand, bounded local classes form a poset under inclusion, and it is clear that we have:

Proposition 2.9 *In any presentable locally cartesian closed ∞ -category, the correspondence between bounded local classes and (equivalence classes of) univalent families is an isomorphism of posets.* \square

Theorem 2.10 *Let $L : \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$ be a accessible locally cartesian localization of a presentable locally cartesian closed ∞ -category \mathcal{P} . Then $p : X \rightarrow S$ is a univalent family in \mathcal{C} if and only if it is also a univalent family in \mathcal{P} .*

Proof. Let T be an object of \mathcal{P} . In the diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(LT, S) & \longrightarrow & \mathcal{C}_{/LT}^{\mathrm{eq}} \\ \simeq \downarrow & & \downarrow \\ \mathrm{Map}_{\mathcal{P}}(T, S) & \longrightarrow & \mathcal{P}_{/T}^{\mathrm{eq}}, \end{array}$$

the left-hand map is an equivalence by adjunction. Since L is a locally cartesian localization, $\mathcal{C}_{/LT} \rightarrow \mathcal{P}_{/T}$ is always fully faithful by Lemma 1.7 and therefore $\mathcal{C}_{/LT}^{\mathrm{eq}} \rightarrow \mathcal{P}_{/T}^{\mathrm{eq}}$ is mono. Hence the top map is mono if and only if the bottom map is mono. \square

2.11 Colocalizations. Although at the moment we do not exploit this in any way, we mention that there is another class of functors which preserve univalent families, the colocalizations.

Proposition 2.12 *Let $R : \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$ be a colocalization (with fully faithful left adjoint $F : \mathcal{C} \rightarrow \mathcal{P}$) between presentable locally cartesian closed ∞ -categories. If $p : X \rightarrow S$ is univalent in \mathcal{P} then $R(p) : R(X) \rightarrow R(S)$ is univalent in \mathcal{C} .*

Proof. For any $T \in \mathcal{C}$, consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(T, RS) & \longrightarrow & \mathcal{C}_{/T}^{\mathrm{eq}} \\ \simeq \uparrow & & \downarrow \\ \mathrm{Map}_{\mathcal{P}}(FT, S) & \longrightarrow & \mathcal{P}_{/FT}^{\mathrm{eq}}, \end{array}$$

where the left-hand vertical map is an equivalence by adjunction, and where the horizontal maps are given by pulling back p and $R(p)$, respectively. That the diagram actually commutes relies on the fact that the unit $\mathrm{id} \Rightarrow RF$ for the adjunction is an equivalence. The right-hand vertical map is mono since F and hence $F : \mathcal{C}_{/T} \rightarrow \mathcal{P}_{/FT}$ are fully faithful. Since $p : X \rightarrow S$ is univalent in \mathcal{P} , the bottom horizontal map is mono, and hence the top horizontal map is also mono, which is to say that $R(p) : R(X) \rightarrow R(S)$ is univalent. \square

3 Factorization systems and truncation

We refer to a factorization system¹³ on an ∞ -category \mathcal{C} , in which \mathcal{E} is the left orthogonal class and \mathcal{F} is the right orthogonal class, by writing $(\mathcal{E}, \mathcal{F})$. We will often require that our factorization systems $(\mathcal{E}, \mathcal{F})$ are stable under basechange; this is automatic for the right orthogonal class \mathcal{F} , so this amounts to saying that pullbacks of arrows in the left orthogonal class \mathcal{E} stay in \mathcal{E} .

Lemma 3.1 *Let \mathcal{C} be a presentable locally cartesian closed ∞ -category \mathcal{C} in which sums are disjoint, and let $(\mathcal{E}, \mathcal{F})$ be a factorization system in \mathcal{C} which is stable under basechange. Then \mathcal{F} is closed under small sums.*

Proof. Let $\{f_i : X_i \rightarrow S_i\}$ be a small set of maps in \mathcal{F} , and let $f : X \rightarrow S$ be their sum in \mathcal{C} . Since sums are universal and disjoint, for each i we have a pullback square

$$\begin{array}{ccc} X_i & \longrightarrow & X \\ f_i \downarrow & \lrcorner & \downarrow f \\ S_i & \longrightarrow & S \end{array}$$

where the horizontal maps are the sum inclusions. Now consider a commutative square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ S' & \longrightarrow & S \end{array} \tag{1}$$

where f' belongs to \mathcal{E} . Pulling back this diagram to S_i yields

$$\begin{array}{ccc} X'_i & \longrightarrow & X_i \\ f'_i \downarrow & & \downarrow f_i \\ S'_i & \longrightarrow & S_i, \end{array}$$

¹³[HTT, Section 5.2.8]

and the sum of all these squares gives back the original square (1), since \mathcal{C} is locally cartesian closed. Because the class \mathcal{E} is stable under basechange, the map f'_i is again in \mathcal{E} , so by orthogonality there is now a diagonal filler u_i for this square. The sum of all the u_i is a filler for the original square, establishing that f is right orthogonal to f' as required. \square

Proposition 3.2 *Let \mathcal{X} be an ∞ -topos and let $(\mathcal{E}, \mathcal{F})$ be a factorization system in \mathcal{X} which is stable under basechange. Then \mathcal{F} is a local class.*

*Proof.*¹⁴ Since \mathcal{F} is closed under sums by the previous lemma, we can apply [HTT 6.2.3.14]: given a pullback diagram

$$\begin{array}{ccc} X_0 & \longrightarrow & X \\ f_0 \downarrow & \lrcorner & \downarrow f \\ Y_0 & \xrightarrow{e} & Y \end{array}$$

in which e is an effective epi, we need to show that if f_0 belongs to \mathcal{F} then already f belongs to \mathcal{F} . Let Y_\bullet be the Čech nerve of $Y_0 \rightarrow Y$, and let X_\bullet be the Čech nerve of $X_0 \rightarrow X$ (the pullback of X to Y_\bullet). At each level n , since \mathcal{F} is stable under basechange, the map $X_n \rightarrow Y_n$ is in \mathcal{F} . We need to check that f is right orthogonal to any map in \mathcal{E} . This condition can be expressed¹⁵ by saying that for any map $p : A \rightarrow B$ in \mathcal{E} , the map given by precomposition with p

$$\mathrm{Map}_{\mathcal{X}/Y}(B, X) \longrightarrow \mathrm{Map}_{\mathcal{X}/Y}(A, X)$$

is a homotopy equivalence. Some abuse of notation is involved here: we assume that the map $p : A \rightarrow B$ is over Y , so as to form the square we need to fill, and the objects B , X and Y are regarded as objects over Y .

Now each of these mapping spaces can be obtained as the totalization of a cosimplicial space (e.g. for B) which in degree n is given by $\mathrm{Map}_{\mathcal{X}/Y_n}(B_n, X_n)$, where A_\bullet and B_\bullet are the objects pulled back to the Čech nerve of $Y_0 \rightarrow Y$. So it is enough to show that for each n , the map

$$\mathrm{Map}_{\mathcal{X}/Y_n}(B_n, X_n) \longrightarrow \mathrm{Map}_{\mathcal{X}/Y_n}(A_n, X_n)$$

is a homotopy equivalence. But this follows from the assumption: we have already remarked that each of the maps f_n is in \mathcal{F} , and since the class \mathcal{E} is assumed to be stable under basechange, also each of the maps p_n is in \mathcal{E} . So by orthogonality of p_n with f_n we do have the required homotopy equivalence. \square

Corollary 3.3 *In an ∞ -topos, the class of n -truncated maps is local ($-2 \leq n < \infty$). The class of hypercomplete maps¹⁶ is also local (and of course the class of all maps is local).*

Proof. Both the factorization system of n -connected maps¹⁷ and n -truncated maps and the factorization system of ∞ -connected maps¹⁸ and hypercomplete maps are stable by [HTT 6.5.1.16]. \square

3.4 \mathcal{Q} -quasi-left-exact localization. A localization $L : \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$ is called \mathcal{Q} -quasi-left-exact, with respect to a class of maps \mathcal{Q} in \mathcal{C} (closed under equivalences), when for each pullback square in \mathcal{P}

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

the natural comparison map $L(X') \rightarrow L(Y') \times_{L(Y)} L(X)$ belongs to \mathcal{Q} .

¹⁴generalizing [HTT 6.5.2.22]

¹⁵dual of [HTT 5.2.8.3]

¹⁶[HTT 6.5.2.21]

¹⁷Following Joyal [12], we define n -connected to mean left orthogonal to n -truncated. In an ∞ -topos this agrees with Lurie's notion of $(n+1)$ -connective.

¹⁸A map is ∞ -connected if it is n -connected for all n .

A map in an ∞ -category \mathcal{C} is called an n -gerbe¹⁹ when it is simultaneously $(n-1)$ -connected and n -truncated. (Intuitively, its only nonzero relative homotopy group is π_n .) Let \mathcal{K}_n denote the class of n -gerbes in \mathcal{C} . Note that \mathcal{K}_n also restricts to a class in the full subcategory $\tau_{\leq n}\mathcal{C} \subseteq \mathcal{C}$ of n -truncated objects.

Lemma 3.5 *For a presentable ∞ -category \mathcal{P} , the truncation functor $\tau_{\leq n} : \mathcal{P} \rightarrow \tau_{\leq n}\mathcal{P}$ is \mathcal{K}_n -quasi-left-exact.*

Proof. For simplicity of notation, set $L = \tau_{\leq n}$. Given a pullback square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ Y' & \longrightarrow & Y, \end{array}$$

we must show that the comparison map $L(Y' \times_Y X) \rightarrow LY' \times_{LY} LX$ belongs to \mathcal{K}_n . Since the involved objects are local, it is enough to show that the map $u : Y' \times_Y X \rightarrow LY' \times_{LY} LX$ is quasi-local, i.e. that applying L to it yields a map in \mathcal{K}_n . But \mathcal{K}_n is defined as the intersection of the classes of $(n-1)$ -connected maps and n -truncated maps, so the class of $(n-1)$ -connected maps is certainly sent to \mathcal{K}_n . It therefore suffices to show that u is $(n-1)$ -connected. But the map u can be factored as a composite of three maps:

$$Y' \times_Y X \xrightarrow{u_1} Y' \times_{LY} X \xrightarrow{u_2} LY' \times_{LY} X \xrightarrow{u_3} LY' \times_{LY} LX.$$

The map u_3 is the pullback of $X \rightarrow LX$, which is n -connected by definition of L , and since connected maps are stable under pullback,²⁰ u_3 is n -connected as well, and in particular $(n-1)$ -connected. The same argument applies to u_2 , which is the pullback of $Y' \rightarrow LY'$. Finally u_1 sits in the diagram

$$\begin{array}{ccc} Y' \times_Y X & \xrightarrow{u_1} & Y' \times_{LY} X \\ \downarrow \lrcorner & & \downarrow \\ Y & \xrightarrow{\delta} & Y \times_{LY} Y. \end{array}$$

Since $Y \rightarrow LY$ is n -connected, the diagonal δ is $(n-1)$ -connected, and hence the pullback u_1 is $(n-1)$ -connected. Altogether $u = u_3 \circ u_2 \circ u_1$ is $(n-1)$ -connected, as desired. \square

Lemma 3.6 *Let \mathcal{U} be a local class in a presentable ∞ -category \mathcal{P} and let $L : \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$ be a \mathcal{Q} -quasi-left-exact localization for a class \mathcal{Q} in \mathcal{C} . Consider a class \mathcal{F} in \mathcal{C} such that $L(\mathcal{U}) \subseteq \mathcal{F}$ and $G(\mathcal{F}) \subseteq \mathcal{U}$ (here G denotes a right adjoint to L). If \mathcal{Q} is left orthogonal to \mathcal{F} then \mathcal{F} is a local class in \mathcal{C} .*

When \mathcal{Q} is the class of equivalences, we are just talking left-exact localization. The lemma implies in this case that left-exact localizations preserve the property that all maps form a local class, and actually the proof is only a slight modification of the proof of this result in [HTT, Proposition 6.1.3.10]. For more general \mathcal{Q} , the class of all maps cannot stay local, but the lemma says that more restricted classes can, provided they are right orthogonal to \mathcal{Q} .

Proof. By [HTT, Lemma 6.1.3.7], we must show that, for each diagram $\sigma : \Lambda_0^2 \rightarrow \mathcal{O}_{\mathcal{C}}$, denoted $g \xleftarrow{\alpha} f \xrightarrow{\beta} h$, in which f, g , and h belong to \mathcal{F} and α and β are cartesian transformations, there is a colimit diagram $\overline{\sigma} : \Lambda_0^{2\triangleright} \rightarrow \mathcal{O}_{\mathcal{C}}$

$$\begin{array}{ccc} f & \xrightarrow{\beta} & h \\ \alpha \downarrow & & \downarrow \alpha' \\ g & \xrightarrow{\beta'} & p \end{array}$$

¹⁹Joyal [12], p.181

²⁰[HTT 6.5.1.16]

such α' and β' are cartesian and p is again in \mathcal{F} . We can assume that $\sigma = L \circ \sigma'$ for some diagram $\sigma' : \Lambda_0^2 \rightarrow \mathcal{O}_{\mathcal{P}}$ equivalent to $G \circ \sigma$. As G is a right adjoint, $G(\alpha)$ and $G(\beta)$ are cartesian, and by assumption $G(f)$, $G(g)$ and $G(h)$ are in \mathcal{U} , so since \mathcal{U} is a local class, there is a colimit diagram $\overline{\sigma}' : \Lambda_0^{2\triangleright} \rightarrow \mathcal{O}_{\mathcal{P}}$

$$\begin{array}{ccc} G(f) & \xrightarrow{G(\beta)} & G(h) \\ G(\alpha) \downarrow & & \downarrow \gamma \\ G(g) & \xrightarrow{\delta} & q \end{array}$$

in which γ and δ are cartesian and q is in \mathcal{U} . Now apply L to get a pushout square

$$\begin{array}{ccc} f & \xrightarrow{\beta} & h \\ \alpha \downarrow & & \downarrow L\gamma \\ g & \xrightarrow{L\delta} & L(q). \end{array}$$

By assumption, $L(q) \in \mathcal{F}$, so it remains to check that $L(\gamma)$ and $L(\delta)$ are cartesian. Let's look at $L(\delta)$: its components are diagrams

$$\begin{array}{ccc} A' & \xrightarrow{L(\delta)_0} & A \\ g \downarrow & & \downarrow L(q) \\ B' & \xrightarrow{L(\delta)_1} & B \end{array}$$

in which g and $L(q)$ are in \mathcal{F} . A priori this square might not be a pullback, but the comparison map u

$$\begin{array}{ccccc} A' & \xrightarrow{u} & B' \times_B A & \longrightarrow & A \\ & \searrow g & \downarrow \pi & \lrcorner & \downarrow Lq \\ & & B' & \xrightarrow{L(\delta)_1} & B \end{array}$$

belongs to \mathcal{Q} . Now consider the class $\overline{\mathcal{F}} = \mathcal{Q}^\perp$. It is stable under pullback, hence the map π belongs to $\overline{\mathcal{F}}$ (since $L(q)$ does), and g also belongs to \mathcal{F} by assumption. It follows²¹ that u also belongs to $\overline{\mathcal{F}}$. Since it also belongs to \mathcal{Q} it must therefore be an equivalence, because the intersection of orthogonal classes is necessarily contained in the class of equivalences. Therefore $L(\delta)$ is cartesian. (In conclusion, although L might not preserve all pullbacks, it does preserve just enough pullbacks to preserve locality). \square

Corollary 3.7 *Suppose the class \mathcal{T}_n of n -truncated maps is local in a presentable locally cartesian closed ∞ -category \mathcal{C} . Then it is also local in $\tau_{\leq n+1}\mathcal{C}$.*

Proof. Let \mathcal{U} be the class of n -truncated maps in \mathcal{C} and let \mathcal{F} be the class of n -truncated maps in $\tau_{\leq n+1}\mathcal{C}$. We have $\mathcal{K}_{n+1} \perp \mathcal{F}$, and the $(n+1)$ -truncation localization $L : \mathcal{C} \rightarrow \tau_{\leq n+1}\mathcal{C}$ is \mathcal{K}_{n+1} -quasi-left-exact by Lemma 3.5. Hence by Lemma 3.6, the class \mathcal{F} is local in $\tau_{\leq n+1}\mathcal{C}$. \square

A presentable locally cartesian closed ∞ -category \mathcal{C} is an n -topos if and only if \mathcal{C} is equivalent to an n -category and the class \mathcal{T}_{n-2} of $(n-2)$ -truncated morphisms in \mathcal{C} is local.²² This fact together with the previous corollary yields the following:

²¹[HTT 5.2.8.6]

²²[HTT 6.4.1.5]

Corollary 3.8 *Let \mathcal{C} be a presentable locally cartesian closed ∞ -category. Then the class \mathcal{T}_n of n -truncated maps in \mathcal{C} is local if and only if $\tau_{\leq n+1}\mathcal{C}$ is an $(n+2)$ -topos.*

Corollary 3.9 *In an n -topos the class \mathcal{T}_k is local for all $-2 \leq k \leq n-2$.*

Note that “ \mathcal{T}_k local” does not in general imply that \mathcal{C} is an n -topos for some $n \geq k+2$: although $\tau_{\leq k+1}\mathcal{C}$ is a $(k+2)$ -topos, \mathcal{C} itself could have higher-dimensional cells that prevents it from being a n -topos for any $n \geq k+2$. Easiest counterexample: \mathcal{T}_{-2} is local always, but not every ∞ -category is an n -topos.

3.10 Stability properties of local classes. A necessary condition for a univalent family to serve as a universe for a type theory, the corresponding (bounded) local class should be stable under the type formations: dependent sums (lowershriek) and dependent products (lowerstar), as well as identity types. Every local class is stable under pullback, by definition. If a local class \mathcal{F} is closed under composition, then it is clearly stable under dependent sums (lowershriek), but only along maps in \mathcal{F} .

Lemma 3.11 *If a local class \mathcal{F} is the right orthogonal class of a basechange-stable factorization system then it is closed under dependent products. Moreover, in this situation, if κ is a strongly inaccessible cardinal such that \mathcal{F}_κ is a local class, then \mathcal{F}_κ is closed under dependent products along maps in \mathcal{F}_κ .*

Proof. Let p be any map, and suppose f is a map in \mathcal{F} . We need to check that $p_*(f)$ belongs to \mathcal{F} too. For this we need to check $e \perp p_*f$ for all maps e in the left class. But we have $e \perp p_*f$ if and only if $p^*e \perp f$, and this last statement is true for all e because p^*e then belongs to the left class by stability. Concerning \mathcal{F}_κ , it remains to observe that if both f and p are relatively κ -compact, then also p_*f is so; this is ensured if κ is strongly inaccessible (see [HTT 5.4.2.9]). \square

Lemma 3.12 *If a local class \mathcal{F} is the right class of a factorization system, then it is closed under taking diagonals: if $f : X \rightarrow Y$ belongs to \mathcal{F} then so does $\delta_f : X \rightarrow X \times_Y X$.*

Proof. If $f : X \rightarrow Y$ belongs to \mathcal{F} then so does the (first) projection $X \times_Y X \rightarrow X$ since it is the pullback of f , and local classes are stable under pullback. But the diagonal is a section of the projection, so since \mathcal{F} is a right class, also the diagonal belongs to \mathcal{F} . \square

4 ∞ -Quasitopoi

In this section we provide significant examples of big universal families outside the realm of ∞ -topoi or n -topoi. Our examples should be viewed as ∞ -quasitopoi, but it is beyond the scope of the present paper to develop the theory of ∞ -quasitopoi, so our treatment is deliberately ad hoc, and aims only at providing some examples. We begin with a definition and a lemma.

4.1 Stable units. A localization $L : \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$ is said to have *stable units*²³ if every pullback of any unit component is inverted by L . That is, in any pullback diagram of the form

$$\begin{array}{ccc} P & \xrightarrow{f^*(\eta_X)} & A \\ \downarrow & \lrcorner & \downarrow f \\ X & \xrightarrow{\eta_X} & LX, \end{array}$$

$L(f^*(\eta_X))$ is an equivalence. Equivalently, L preserves basechange along unit components.

Lemma 4.2 *If a localization $L : \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$ has stable units then it preserves pullbacks over any object in \mathcal{C} (and conversely). In particular, a localization with stable units is locally cartesian.*

²³The terminology (in the ordinary-category case) is due to Cassidy–Hébert–Kelly [5].

Proof. The statement is that any pullback square of the form

$$\begin{array}{ccc} P & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow f \\ B & \xrightarrow{g} & LX \end{array}$$

is preserved by L . But this square decomposes into four pullback squares like this:

$$\begin{array}{ccccc} P & \longrightarrow & \cdot & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \eta_A \\ \cdot & \longrightarrow & \cdot & \longrightarrow & LA \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow Lf \\ B & \xrightarrow{\eta_B} & LB & \xrightarrow{Lg} & LX \end{array}$$

The right-hand bottom pullback square is already inside \mathcal{C} so it is clearly preserved by L ; the other squares are pullbacks along units, so they are also preserved by L , by assumption. \square

4.3 ∞ -Quasitopoi. There are several possibilities for defining a notion of ∞ -quasitopos, corresponding to the different possible characterizations of (Grothendieck) quasitopoi in the classical case, recently provided by Garner and Lack [9], and one may expect these definitions to be equivalent. Here we take the viewpoint that a quasitopos is the full subcategory of a presheaf category consisting of those presheaves which are separated with respect to an accessible left-exact localization. The classical notion relies only on the (epi, mono) factorization system. In the ∞ -case, one may envisage a distinct notion of ∞ -quasitopos for each of the factorization systems (n -connected, n -truncated), and indeed one can relate the definition to more general factorization systems, as we now proceed to do.

Let \mathcal{P} be a presheaf ∞ -category (or any ∞ -topos), and let $L : \mathcal{P} \rightarrow \mathcal{C} \subseteq \mathcal{P}$ be an accessible left-exact localization, with right adjoint (inclusion) G and unit η . Let $(\mathcal{E}, \mathcal{F})$ be a (orthogonal) factorization system in \mathcal{P} , stable under basechange and with the property that L preserves both classes, i.e. if f belongs to \mathcal{F} (resp. \mathcal{E}) then also GLf belongs to \mathcal{F} (resp. \mathcal{E}). We say that an object $Q \in \mathcal{P}$ is \mathcal{F} -separated with respect to L if $\eta_Q : Q \rightarrow GLQ$ belongs to \mathcal{F} . Let $\mathcal{Q} \subseteq \mathcal{P}$ denote the full subcategory consisting of the \mathcal{F} -separated objects. We define an ∞ -quasitopos to be a presentable ∞ -category which arises in this way; that is, as the ∞ -category of \mathcal{F} -separated objects in a presheaf ∞ -category equipped with an accessible left-exact localization L and a basechange-stable factorization system $(\mathcal{E}, \mathcal{F})$ which is preserved by L .

An important example of this situation is the factorization system (n -connected, n -truncated), which is stable by [HTT 6.5.1.16]: any left-exact localization preserves n -truncated maps,²⁴ and it also preserves n -connected maps if both domain and codomain are ∞ -topoi (so that L is the left-adjoint part of a geometric morphism).²⁵

The following theorem is an ∞ -version of Proposition 3.3 and Corollary 3.4 of Garner and Lack [9]. Our proof draws upon their arguments.

Theorem 4.4 *Let \mathcal{X} be an ∞ -topos, presented as a left-exact localization $L : \mathcal{P} \rightarrow \mathcal{X} \subseteq \mathcal{P}$ of an ∞ -topos \mathcal{P} equipped with a basechange-stable factorization system $(\mathcal{E}, \mathcal{F})$ which is preserved by L , and let $\mathcal{Q} \subseteq \mathcal{P}$ denote the full subcategory of \mathcal{F} -separated objects. Then the inclusion functors $\overline{G} : \mathcal{X} \rightarrow \mathcal{Q}$ and $G' : \mathcal{Q} \rightarrow \mathcal{P}$ admit left adjoints $\overline{L} : \mathcal{Q} \rightarrow \mathcal{X}$ and $L' : \mathcal{P} \rightarrow \mathcal{Q}$, respectively, such that \overline{L} is left-exact and L' preserves the factorization system $(\mathcal{E}, \mathcal{F})$ and has stable units (so in particular L' is locally cartesian).*

²⁴[HTT, Prop. 5.5.6.16]

²⁵[HTT, Prop. 6.5.1.16 (4)]

The proof follows the following few remarks.

4.5 Univalence in ∞ -quasitopoi. The theorem says that any ∞ -quasitopos arises as an accessible localization with stable units and which preserves the class \mathcal{F} . In the classical case, with the (epi-mono) factorization system, this is one of the possible characterizations of quasitopoi [9]. In particular, an ∞ -quasitopos is presentable and locally cartesian closed.

The localization $\overline{\mathcal{L}} : \mathcal{Q} \rightarrow \mathcal{C} \subseteq \mathcal{Q}$ now satisfies the conditions of Theorem 2.10. Hence:

Corollary 4.6 *Let \mathcal{Q} be an ∞ -quasitopos containing an ∞ -topos \mathcal{X} as in Theorem 4.4. Then any univalent family in \mathcal{X} is also a univalent family in \mathcal{Q} .*

In particular, since \mathcal{X} is an ∞ -topos, for each regular cardinal κ there is a universal (and hence univalent) family classifying all κ -compact maps, which therefore also constitutes a univalent family in \mathcal{Q} , although it no longer classifies all maps. The maps in \mathcal{Q} obtained as pullbacks of this univalent family form a bounded local class, by Proposition 2.5. Presumably there should be a direct characterization of this local class in terms of some notion of strong map, depending on the factorization system $(\mathcal{E}, \mathcal{F})$, just as in the classical case the local class consists of strong monos.

4.7 Proof of Theorem 4.4. Step 1: G' has a left adjoint L' . To establish that G' has a left adjoint, we must show that for every $X \in \mathcal{P}$ the under ∞ -category $\mathcal{Q}_{X/}$ has an initial object;²⁶ then the left adjoint L' will be given on objects by sending X to the codomain of this initial object. Consider the $(\mathcal{E}, \mathcal{F})$ -factorization of $\eta : X \rightarrow GLX$:

$$\begin{array}{ccc} X & \xrightarrow{\eta} & GLX \\ & \searrow \eta' & \nearrow \lambda \\ & X' & \end{array} \quad (2)$$

Since λ belongs to \mathcal{F} , the object X' is in \mathcal{Q} . We claim that $\eta' : X \rightarrow X'$ is an initial object of $\mathcal{Q}_{X/}$. Let $f : X \rightarrow Y'$ be another object of $\mathcal{Q}_{X/}$. We must show that the fiber $\text{Map}_{X/}(X', Y')$ of the map

$$\text{Map}(X', Y') \longrightarrow \text{Map}(X, Y')$$

given by precomposition by η is contractible. Consider the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y' & & \\ \eta' \downarrow & & \downarrow \eta & & \\ X' & \xrightarrow{\lambda} & GLX & \xrightarrow{GLf} & GLY' \end{array} \quad (3)$$

Here the left-hand vertical map η' belongs to \mathcal{E} by construction, and the right-hand vertical map η belongs to \mathcal{F} because $Y' \in \mathcal{Q}$. Since these are orthogonal in \mathcal{P} , there is a unique diagonal filler. More formally, the orthogonality $\eta' \perp \eta$ is expressed by the upper square being a pullback in the diagram

$$\begin{array}{ccc} \text{Map}_{\mathcal{P}}(X', Y') & \longrightarrow & \text{Map}_{\mathcal{P}}(X, Y') \\ \downarrow \perp & & \downarrow \\ \text{Map}_{\mathcal{P}}(X', GLY') & \longrightarrow & \text{Map}_{\mathcal{P}}(X, GLY') \\ \downarrow \simeq & & \downarrow \simeq \\ \text{Map}_{\mathcal{X}}(LX', LY') & \longrightarrow & \text{Map}_{\mathcal{X}}(LX, LY') \end{array} \quad (4)$$

²⁶This is a standard result for ordinary categories. For ∞ -categories, a proof can be found in [10].

where the horizontal maps are pre-composition with η' , and the vertical maps of the top square are post-composition with η . (The bottom square is just adjunction.) We now claim that $L\eta' : LX \rightarrow LX'$ is an equivalence. Indeed, apply L to the triangle (2) to obtain

$$\begin{array}{ccc} LX & \xrightarrow{\simeq} & LGLX \\ & \searrow L\eta' & \nearrow L\lambda \\ & LX' & \end{array}$$

Since L preserves the class of maps \mathcal{F} , the map $L\lambda$ is in \mathcal{F} , and hence $L\eta'$ is too. But $L\eta'$ is also in \mathcal{E} , since L also preserves the class \mathcal{E} . Hence $L\eta'$ is an equivalence (and $L\lambda$ is therefore too). It follows that the bottom map in (4) is an equivalence. Hence the other horizontal maps are equivalences too.

Step 2: L' preserves the both the classes \mathcal{E} and \mathcal{F} . Indeed, for a map $f : X \rightarrow Y$ in \mathcal{P} , consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{\eta'} & L'X & \xrightarrow{\lambda} & LX \\ f \downarrow & & L'f \downarrow & & \downarrow Lf \\ Y & \xrightarrow{\eta'} & L'Y & \xrightarrow{\lambda} & LY. \end{array}$$

If f is in \mathcal{E} , then it follows that $L'f$ is in \mathcal{E} because so are the two instances of η' . On the other hand if f is in \mathcal{F} , then by assumption on L , also Lf is in \mathcal{F} . But so are the two instances of λ , so we conclude that $L'f$ is in \mathcal{F} .

Step 3: L' preserves pullbacks along components of η . Consider a pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{\eta} & LX \end{array}$$

Upon applying L' and L , we get this diagram

$$\begin{array}{ccccccc} X \times_{LX} A & \xrightarrow{\eta'} & L'(X \times_{LX} A) & \xrightarrow{\lambda} & L(X \times_{LX} A) \\ \downarrow = & & \downarrow \pi' & & \downarrow \pi \\ X \times_{LX} A & \xrightarrow{\eta' \times \text{id}} & L'X \times_{LX} A & \xrightarrow{\text{id} \times \eta'} & L'X \times_{LX} L'A & \xrightarrow{\lambda \times \text{id}} & LX \times_{LX} L'A & \xrightarrow{\text{id} \times \lambda} & LX \times_{LX} LA. \end{array}$$

Here π and π' are the canonical comparison maps; we must show that π' is an equivalence. From the right-hand square we conclude that π' is in \mathcal{F} : indeed, any λ -map is in \mathcal{F} , and so are pullbacks of λ -maps, such as the bottom two maps in that square. But π is an equivalence since L is left-exact, so the remaining side π' must be in \mathcal{F} too.²⁷ From the left-hand square we conclude that π' is in \mathcal{E} : indeed, η' -maps belong to \mathcal{E} , and by basechange stability of \mathcal{E} , so do pullbacks of η' -maps, such as the bottom two maps in that square; hence the remaining side π' must be in \mathcal{E} too.²⁸ Altogether π' is therefore an equivalence, as desired.

Step 4: Any map of the form $X \times_{L'X} A \rightarrow X \times_{LX} A$ is in \mathcal{F} . Indeed, it sits in the pullback diagram

$$\begin{array}{ccc} X \times_{L'X} A & \longrightarrow & L'X \times_{L'X} L'X \\ \downarrow \lrcorner & & \downarrow \delta_\lambda \\ X \times_{LX} A & \longrightarrow & L'X \times_{LX} L'X \end{array}$$

²⁷[HTT 5.2.8.6 (3)]

²⁸[HTT 5.2.8.6 (4)]

where the right-hand vertical map is equivalent to the diagonal of $\lambda : L'X \rightarrow LX$. Since λ belongs to \mathcal{F} , by Lemma 3.12 also δ_λ belongs to \mathcal{F} , and hence also its pullback, as claimed.

Step 5: L' preserves pullbacks along components of η' , i.e. L' has stable units. Applying L' to the pullback diagram

$$\begin{array}{ccc} P & \xrightarrow{q} & A \\ p \downarrow & \lrcorner & \downarrow u \\ X & \xrightarrow{\eta'} & L'X \end{array}$$

gives

$$\begin{array}{ccccc} & & L'q & & \\ & \curvearrowright & & \curvearrowright & \\ L'P & \xrightarrow{a} & L'X \times_{L'L'X} L'A & \xrightarrow{\simeq} & L'A \\ & \searrow L'p & \downarrow \lrcorner & & \downarrow L'u \\ & & L'X & \xrightarrow[L'\eta']{\simeq} & L'L'X. \end{array}$$

The map a also sits in the diagram

$$\begin{array}{ccc} L'P & \xrightarrow{a} & L'X \times_{L'L'X} L'A \\ \downarrow & & \downarrow \\ L'(X \times_{LX} A) & \xrightarrow{\simeq} & L'X \times_{LX} L'A. \end{array}$$

Here the right-hand vertical map is an example of the situation in Step 4, so it belongs to \mathcal{F} . The left-hand vertical map is also such an example but with L' applied to it. Since L' preserves the class \mathcal{F} by Step 2, it is again in \mathcal{F} . The bottom map is an equivalence since L' preserves pullbacks along components of η , by Step 3. It follows from this that also a is in \mathcal{F} . Hence $L'q$ is in \mathcal{F} . On the other hand, since η' is in \mathcal{E} , by pullback stability of this class, also q is in \mathcal{E} , and since L' preserves the class \mathcal{E} by Step 2, also $L'q$ is in \mathcal{E} , so altogether $L'q$ is an equivalence. Hence a is an equivalence, which is to say that L' preserves pullbacks along η' .

Step 6: It is easy to see that L restricts to a functor $\overline{L} : \mathcal{Q} \rightarrow \mathcal{X}$ right adjoint to $\overline{G} : \mathcal{X} \rightarrow \mathcal{Q}$. \square

5 Bundles and connected univalent families

In this section, we study the univalent family $p : X \rightarrow S$ associated to a given object F of an ∞ -topos \mathcal{X} . Specifically, we show that the universal F -bundle over $B\mathbf{Aut}(F)$ is univalent; these provide examples of univalent families with connected base (unlike the universal univalent families, which have many connected components). Finally, we consider some sporadic examples, outside the realm of ∞ -topoi.

5.1 Bundles and automorphisms. Recall that an F -bundle (or a bundle with fiber F) in an ∞ -topos \mathcal{X} is a map $X \rightarrow S$ for which there exist an effective epi $T \rightarrow S$ such that the pullback $Y = X \times_S T \rightarrow T$ is equivalent (as an object over T) to the trivial F -bundle, the projection $F \times T \rightarrow T$. Also recall that F defines a sheaf $\mathbf{Aut}(F)$ of automorphisms of F , whose T -points is the space $\mathbf{Aut}_{/T}(F \times T)$ of automorphisms of $F \times T$ over T . Finally, given an object X of \mathcal{X} equipped with a (right) action $G \rightarrow \mathbf{Aut}(X)$ by a group object G of \mathcal{X} , we write X/G for the quotient of this action, that is, the geometric realization of the simplicial object which in degree n is $X \times G^n$.

Lemma 5.2 *For F an object of an ∞ -topos, the class of F -bundles is a bounded local class, and the corresponding univalent family (classifying object for F -bundles) is*

$$F/\mathbf{Aut}(F) \longrightarrow */\mathbf{Aut}(F) = B\mathbf{Aut}(F).$$

Proof. We apply the criterion of [HTT, Prop. 6.2.3.14]. It is clear that F -bundles are stable under pullbacks and sums, so it remains to check that if

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & \lrcorner & \downarrow f \\ Y' & \xrightarrow{e} & Y \end{array}$$

is a pullback along an effective epi e , and if f' is an F -bundle, then also f is an F -bundle. But to say that f' is an F -bundle means that there exists an effective epi $Y'' \rightarrow Y'$ that trivializes it. That is to say that the pullback of f' to Y'' is a trivial F -bundle. But then the composite effective epi $Y'' \rightarrow Y' \rightarrow Y$ also trivializes f , which is therefore an F -bundle.

To check that $F/\underline{\mathbf{Aut}}(F) \rightarrow */\underline{\mathbf{Aut}}(F)$ is the universal family for F -bundles, it suffices to show that the sheaf \mathcal{B}_F which associates to S the space of F -bundles $X \rightarrow S$ is represented by $B\underline{\mathbf{Aut}}(F)$. Since any F -bundle is locally trivial, we conclude that there exists an effective epi $* \rightarrow \mathcal{B}_F$. Moreover, the pullback $* \times_{\mathcal{B}_F} *$ is equivalent to the group object $\underline{\mathbf{Aut}}(F)$ of automorphisms of the trivial F -bundle $F \rightarrow *$. Hence \mathcal{B}_F is equivalent to the quotient of the action of $\underline{\mathbf{Aut}}(F)$ on $*$, the definition of $B\underline{\mathbf{Aut}}(F)$. \square

5.3 Sections and pointed objects. Typically, given a family $p : X \rightarrow S$ and a map $f : T \rightarrow S$, the pullbacks $f^*p : f^*X \rightarrow T$ do not admit sections. They do, however, if f happens to factor through p itself. In particular, the pullback $p^*p : p^*X \rightarrow X$, which is none other than the projection $\pi : X \times_S X \rightarrow X$, admits a diagonal section $\delta : X \rightarrow X \times_S X$. It follows that we have a commutative square

$$\begin{array}{ccc} \mathrm{Map}(T, X) & \longrightarrow & \mathcal{C}_{/T*}^{\mathrm{eq}} \\ \downarrow & & \downarrow \\ \mathrm{Map}(T, S) & \longrightarrow & \mathcal{C}_{/T}^{\mathrm{eq}} \end{array} \tag{5}$$

in which the upper right-hand corner is the (maximal subgroupoid of) the ∞ -category of objects over and under T , i.e. the ∞ -category $(\mathcal{C}_{/T})_{\mathrm{id}_T}^{\mathrm{eq}}$ of pointed objects of $\mathcal{C}_{/T}$, and the top horizontal map sends $f : T \rightarrow S$ to the pullback of $X \xrightarrow{\delta} X \times_S X \xrightarrow{\pi} X$ along f .

Proposition 5.4 *Let \mathcal{C} be a locally cartesian closed ∞ -category and let $p : X \rightarrow S$ be a univalent family in \mathcal{C} . Then, for any object T of \mathcal{C} , the map*

$$\mathrm{Map}(T, X) \longrightarrow \mathcal{C}_{/T*}^{\mathrm{eq}}$$

described above is a monomorphism.

Proof. Choose a basepoint $f : T \rightarrow S$ of $\mathrm{Map}(T, S)$ and write f^*X for its image in $\mathcal{C}_{/T}^{\mathrm{eq}}$. We have the commutative diagram

$$\begin{array}{ccccc} \mathrm{Map}_{/S}(f, p) & \xrightarrow{\cong} & \mathrm{Map}_{/T}(\mathrm{id}_T, f^*p) & & \\ \downarrow & & \downarrow & & \\ \mathrm{Map}(T, X) & \longrightarrow & \mathcal{C}_{/T*}^{\mathrm{eq}} & \longrightarrow & \mathcal{C}_{/T*} \\ \downarrow & & \downarrow \lrcorner & & \downarrow \\ \mathrm{Map}(T, S) & \longrightarrow & \mathcal{C}_{/T}^{\mathrm{eq}} & \longrightarrow & \mathcal{C}_{/T}. \end{array}$$

The outer vertical maps form fiber sequences (over f and f^*p respectively). The right-hand bottom square is a pullback square since its right-hand side is conservative. The top vertical map of the diagram is an

equivalence by adjunction, and since this holds for any object $f \in \text{Map}(T, S)$, this implies that the bottom outer square is a pullback. Hence the left-hand bottom square is a pullback too. Finally since p is univalent, the map $\text{Map}(T, S) \rightarrow \mathcal{C}_{/T}^{\text{eq}}$ is mono, and hence the map $\text{Map}(T, X) \rightarrow \mathcal{C}_{/T*}^{\text{eq}}$ is mono too, as asserted. \square

5.5 n -Truncated univalent families in $(n+1)$ -topoi. It is tempting to suppose that any univalent family in an $(n+1)$ -topos must belong to the class of $(n-1)$ -truncated morphisms. Indeed, an $(n+1)$ -category is an $(n+1)$ -topos if and only if it is presentable, locally cartesian closed, and the $(n-1)$ -truncated maps form a local class. While this is true for $n = \infty$, this fails in general for $n < \infty$.

Consider, for instance, the case $\mathcal{C} = \tau_{\leq n}\mathcal{S}$, the $(n+1)$ -topos of n -truncated spaces. By 5.2, any n -truncated space F is potentially gives rise to a univalent family

$$p : F / \text{Aut}(F) \longrightarrow * / \text{Aut}(F) \simeq B \text{Aut}(F)$$

classifying morphisms with fiber F ; however, $B \text{Aut}(F)$ is typically not n -truncated, so this map need not live in \mathcal{C} . (Of course, if F is $(n-1)$ -truncated, then $\text{Aut}(F)$ is also $(n-1)$ -truncated and consequently $B \text{Aut}(F)$ is n -truncated.) Hence counterexamples arise when F is only n -truncated but $\text{Aut}(F)$ is $(n-1)$ -truncated.

In particular, if G is a discrete group and $F = BG$, which we regard as a pointed connected space, then there is a fiber sequence

$$\text{Aut}_*(BG) \longrightarrow \text{Aut}(BG) \longrightarrow BG$$

in which, by the equivalence between pointed connected objects and group objects of an ∞ -topos,²⁹ we may identify the fiber $\text{Aut}_*(BG)$ with the space of group automorphisms of G , which is discrete since G is discrete. Using the long exact sequence

$$* \cong \pi_1 \text{Aut}_*(BG) \longrightarrow \pi_1 \text{Aut}(BG) \longrightarrow \pi_1 BG \cong G$$

we calculate that $\pi_1 \text{Aut}(BG)$ is isomorphic to the kernel of the conjugation action map

$$G \longrightarrow \text{Aut}_{\text{Gp}}(G).$$

Hence $\pi_1 \text{Aut}(BG) \cong Z(G)$, the center of G , and we conclude that $\text{Aut}(BG)$ is discrete if and only if G has trivial center. In this case, $\text{Aut}(BG) \simeq \pi_0 \text{Aut}(BG) \cong \text{Out}(G)$, and we obtain a univalent family

$$p : BG / \text{Out}(G) \longrightarrow B\text{Out}(G)$$

in $\text{Gpd} \simeq \tau_{\leq 1}\mathcal{S}$ which is not 0-truncated. Note that there are infinite families of groups with trivial center, e.g. the dihedral groups D_n for n odd, the symmetric groups Σ_n for $n > 2$, or any simple group.

5.6 Some univalent families outside topoi. An interesting example of a presentable locally cartesian closed ∞ -category \mathcal{C} which is not an n -topos for any $0 \leq n \leq \infty$ is the \mathbb{A}^1 -localization of the ∞ -topos $\text{Shv}_{\text{Nis}}(\text{Sm}_S)$ of Nisnevich sheaves of spaces on smooth S -schemes of Morel-Voevodsky [15]. Here S denotes a fixed base scheme, which we take to be integral, Noetherian, and of finite Krull dimension (e.g. $S = \text{Spec}(k)$ for k a field). Recall that a Nisnevich sheaf of groups G on Sm_S is said to be *strongly \mathbb{A}^1 -invariant* if, in the ∞ -topos $\text{Shv}_{\text{Nis}}(\text{Sm}_S)$, its classifying space BG is \mathbb{A}^1 -local.

Proposition 5.7 [17, Remark 3.5] *The \mathbb{A}^1 -localization $\mathcal{C} = \text{Shv}_{\text{Nis}}^{\mathbb{A}^1}(\text{Sm}_S)$ of $\text{Shv}_{\text{Nis}}(\text{Sm}_S)$ is a presentable locally cartesian ∞ -category which is not an n -topos for any $0 \leq n \leq \infty$.*

Proof. Given a map of sheaves $f : U \rightarrow T$ and a smooth S -scheme $X \rightarrow T$ over T , we have

$$f^*(X \times \mathbb{A}^1) \cong (f^*X) \times \mathbb{A}^1,$$

which is to say that basechange preserves the generating \mathbb{A}^1 -local equivalences. Hence the localization is locally cartesian by Proposition 2.5. Clearly \mathcal{C} is not an n -topos for any $n < \infty$ since it contains objects

²⁹[HTT 7.2.2.11].

which are not $(n - 1)$ -truncated, so we must show that \mathcal{C} is not an ∞ -topos. This follows from the fact that there exist group objects G of \mathcal{C} which are not strongly \mathbb{A}^1 -local, so that the natural map $G \rightarrow \Omega L^{\mathbb{A}^1} BG$ cannot be an equivalence. \square

5.8 Univalent families in motivic homotopy theory. Let F be an \mathbb{A}^1 -local object of $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S)$ such that $G = \underline{\mathrm{Aut}}(F)$ is a strongly \mathbb{A}^1 -invariant group object. Then $p : F/G \rightarrow */G \simeq BG$ is a univalent family in $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S)$ such that the source and target are \mathbb{A}^1 -local objects, since F/G sits in a fibration $F \rightarrow F/G \rightarrow BG$ and is therefore also \mathbb{A}^1 -local.

Lemma 5.9 *Let G be a strongly \mathbb{A}^1 -invariant group over S with the property that $\underline{\mathrm{Aut}}_*(G)$ is also strongly \mathbb{A}^1 -invariant. Then $G/\underline{\mathrm{Aut}}(G) \rightarrow B\underline{\mathrm{Aut}}(G)$ is a univalent family in $\mathrm{Shv}_{\mathrm{Nis}}^{\mathbb{A}^1}(\mathrm{Sm}_S)$.*

Example 5.10 Take for instance $F = \mathbb{G}_m$, the multiplicative group over S . Since we have assumed S integral, there are no nonconstant units of \mathbb{A}^1 , so that \mathbb{G}_m is \mathbb{A}^1 -local. To calculate $\underline{\mathrm{Aut}}(\mathbb{G}_m)$, we use the fact that

$$\mathbb{G}_m[2] \cong \underline{\mathrm{Aut}}_*(\mathbb{G}_m) \rightarrow \underline{\mathrm{Aut}}(\mathbb{G}_m) \rightarrow \mathbb{G}_m$$

is a fibration sequence, so it deloops to a fibration sequence

$$B\mathbb{G}_m[2] \rightarrow B\underline{\mathrm{Aut}}(\mathbb{G}_m) \rightarrow B\mathbb{G}_m$$

(here $\mathbb{G}_m[2]$ denotes the 2-torsion subgroup of \mathbb{G}_m , which is a finite discrete group scheme over S). Since both $B\mathbb{G}_m$ and $B\mathbb{G}_m[2]$ are \mathbb{A}^1 -local, we obtain a univalent family by the lemma.

Example 5.11 One obtains a similar result for elliptic curves C , which are \mathbb{A}^1 -local since any map $\mathbb{A}_T^1 \cong \mathbb{A}^1 \times_S T \rightarrow C$ over S induces a map $\mathbb{A}_T^1 \rightarrow C_T$ over T and thus can be completed to a map $\mathbb{P}_T^1 \rightarrow C_T$ over T . But C_T has no rational curves, so any such map must be constant. Moreover, $\underline{\mathrm{Aut}}_*(C)$ is a finite group S -scheme, as the moduli stack $\mathcal{M}_{1,1}$ of genus one curves with one marked point is Deligne-Mumford. It follows that $\underline{\mathrm{Aut}}_*(C)$ is also strongly \mathbb{A}^1 -invariant.

6 Univalence in type-theoretic model categories

The univalence property is a homotopy invariant notion and is therefore independent of the particular features of a model. However, in order to get a literal interpretation of type theory in homotopy theory, certain strictness features are required [16]; these are not homotopy invariant and are therefore features manifest only on the level of the model category. In this section we show that the standard Quillen model category \mathcal{M} associated to a presentable locally cartesian closed ∞ -category \mathcal{C} is a “type-theoretic model category,” and that any univalent family in \mathcal{C} lifts to a univalent fibration in \mathcal{M} .

6.1 Type-theoretic model categories. By a *type-theoretic model category* we understand a proper combinatorial model category in which the cofibrations are exactly the monomorphisms, and whose underlying category is locally cartesian closed. These conditions, which are natural from the viewpoint of ∞ -categories, are slightly stronger than the five axioms proposed by Shulman [16] (which in turn are more restrictive than the notion of logical model category of Arndt and Kapulkin [1]), but if we add “combinatorial” to his axioms, then the two notions are equivalent in the sense that any ∞ -category presented by a model category of one sort is also presented by one of the other; this is a consequence of Theorem 6.7 below. The relationship between proper model categories and locally cartesian closed ∞ -categories has been considered recently in [7], where D.-C. Cisinski outlined a proof of Theorem 6.7 below. Our proof is somewhat different from his.

Lemma 6.2 *Let \mathcal{M} be model category in which the cofibrations are the monos. If for any fibration $f : T \rightarrow S$ between fibrant objects, the pullback functor $f^* : \mathcal{M}_{/S} \rightarrow \mathcal{M}_{/T}$ preserves trivial cofibrations, then \mathcal{M} is right proper.*

*Proof.*³⁰ Let $f : T \rightarrow S$ be any fibration. Consider a fibrant replacement $f' : T' \rightarrow S'$ (again a fibration), and let g denote the pullback of f' to S . We have a diagram

$$\begin{array}{ccccc}
 & & f & & \\
 & \curvearrowright & & \curvearrowright & \\
 T & \xrightarrow{j} & T' \times_{S'} S & \xrightarrow{g} & S \\
 & \searrow q_T & \downarrow m & \lrcorner & \downarrow q_S \\
 & & T' & \xrightarrow{f'} & S'
 \end{array}$$

Here q_T and q_S are trivial cofibrations, and $f' : T' \rightarrow S'$ is a fibration between fibrant objects. Since f'^* preserves trivial cofibrations, also m is a trivial cofibration. Since q_T and m are monos, j is again a mono. Since q_T and m are weak equivalences, j is again a weak equivalence, so j is a trivial cofibration. Now consider the top triangle, written as

$$\begin{array}{ccc}
 T & \xrightarrow{=} & T \\
 j \downarrow & & \downarrow f \\
 T' \times_{S'} S & \xrightarrow{g} & S
 \end{array}$$

Since j is a trivial cofibration and f is a fibration, there is a diagonal filler, so we conclude that j has a retraction. Lastly, consider an arbitrary weak equivalence $w : A \rightarrow S$

$$\begin{array}{ccccc}
 \cdot & \xrightarrow{i} & \cdot & \xrightarrow{\quad} & A \\
 u \downarrow & \lrcorner & \downarrow v & \lrcorner & \downarrow w \\
 T & \xrightarrow{j} & T' \times_{S'} S & \xrightarrow{g} & S \\
 & \searrow q_T & \downarrow m & \lrcorner & \downarrow q_S \\
 & & T' & \xrightarrow{f'} & S'
 \end{array}$$

We want to show that u is a weak equivalence. The point is that i also admits a retraction, so that u is a retract of v . But v is a weak equivalence since w is, because pullback along f' preserves trivial cofibrations by assumption, and trivial fibrations as does any pullback functor. Therefore u is a weak equivalence. \square

6.3 Underlying ∞ -categories. Every (possibly large) category \mathcal{M} equipped with a collection of weak equivalences W has an underlying (possibly very large) ∞ -category \mathcal{C} , determined up to contractible ambiguity, obtained³¹ by first passing to the (strict) nerve $N(\mathcal{M})$ and then inverting the weak equivalences in the ∞ -category of (possibly very large) ∞ -categories; that is, $\mathcal{C} = N(\mathcal{M})[W^{-1}]$. This last step amounts to taking a fibrant replacement for the pair $(N(\mathcal{M}), W)$ in the category of marked simplicial sets (see [HTT, Section 3.1]), where W is now considered as a class of marked edges of $N(\mathcal{M})$ (which we may assume contains all degenerate edges).

Now suppose that \mathcal{M} is a model category with subcategory W of weak equivalences. Then for every fibrant object $S \in \mathcal{M}$, the slice category $\mathcal{M}_{/S}$ with its induced model structure (and class of weak equivalences denoted W_S) has the correct homotopy type, in the sense that its underlying ∞ -category $N(\mathcal{M}_{/S})[W_S^{-1}]$ is equivalent to the slice ∞ -category $(N(\mathcal{M})[W^{-1}])_{/S}$.

If V is a strongly saturated³² class of maps in \mathcal{M} (assumed to contain W), we can consider the left Bousfield localization $\mathcal{M} \rightarrow V^{-1}\mathcal{M}$. The underlying ∞ -category of $V^{-1}\mathcal{M}$ is naturally equivalent to the

³⁰Cisinski [7] derives this fact (in the case where \mathcal{M} is a topos) from a fancier result concerning localizers [6, Thm. 4.8]. We think the present elementary proof deserves to be known.

³¹[HA 1.3.4.15].

³²I.e. weakly saturated as in [HTT A.1.2.2] plus closure under 2-for-3; see also [HTT 5.5.4.5].

localization of $N(\mathcal{M})[W^{-1}]$ along (the saturation of) V :

$$N(V^{-1}\mathcal{M})[W^{-1}] \simeq V^{-1}N(\mathcal{M})[W^{-1}].$$

Finally, on both the model level and the ∞ -level, these localizations are compatible with slicing: for each fibrant object $S \in \mathcal{M}$ there is induced a Bousfield localization $\mathcal{M}_{/S} \rightarrow V_S^{-1}\mathcal{M}_{/S}$ whose underlying ∞ -functor is $\mathcal{C}_{/S} \rightarrow V_S^{-1}\mathcal{C}_{/S} \simeq (V^{-1}\mathcal{C})_{/S}$.

Proposition 6.4 *Let \mathcal{M} be a type-theoretic model category. Then the underlying ∞ -category $\mathcal{C} = N(\mathcal{M})[W^{-1}]$ of \mathcal{M} is presentable and locally cartesian closed.*

Proof. Since \mathcal{M} is combinatorial, \mathcal{C} is presentable.³³ Let $f : T \rightarrow S$ be an arrow of \mathcal{C} , which (replacing f by an equivalent arrow if necessary) we may assume is a fibration between fibrant objects in \mathcal{M} . Then we have equivalences

$$\mathcal{C}_{/S} \simeq N(\mathcal{M}_{/S})[W_S^{-1}],$$

and similarly for T . Since by hypothesis, $f^* : \mathcal{M}_{/S} \rightarrow \mathcal{M}_{/T}$ is a left Quillen functor, it follows³⁴ that the induced functor

$$f^* : \mathcal{C}_{/S} \simeq N(\mathcal{M}_{/S})[W_S^{-1}] \longrightarrow N(\mathcal{M}_{/T})[W_T^{-1}] \simeq \mathcal{C}_{/T}$$

admits a right adjoint f_* . □

The following result can be regarded as a converse result, stated in a relative setting.

Proposition 6.5 *Let \mathcal{M} be a type-theoretic model category, let V be a strongly saturated class of maps of \mathcal{M} of small generation, and let $\mathcal{M} \rightarrow V^{-1}\mathcal{M}$ be the left Bousfield localization of \mathcal{M} along V . If the induced localization on the level of ∞ -categories*

$$L : N(\mathcal{M})[W^{-1}] \longrightarrow V^{-1}N(\mathcal{M})[W^{-1}]$$

is locally cartesian then $V^{-1}\mathcal{M}$ is again a type-theoretic model category.

Proof. To show that $V^{-1}\mathcal{M}$ is type-theoretic, the only non-trivial point is to establish that it is right proper. By Lemma 6.2, it is enough to show that f^* preserves local equivalences whenever $f : T \rightarrow S$ is a V -local fibration between V -local fibrant objects. Since S is a V -local fibrant object, we have equivalences

$$\mathcal{C}_{/S} \simeq N(\mathcal{M}_{/S})[W_S^{-1}]$$

and

$$V_S^{-1}\mathcal{C}_{/S} \simeq N(V_S^{-1}(\mathcal{M}_{/S}))[W_S^{-1}],$$

as well as the corresponding equivalences involving T . There is an induced pullback functor

$$f^* : V_S^{-1}\mathcal{C}_{/S} \simeq (V^{-1}\mathcal{C})_{/S} \longrightarrow (V^{-1}\mathcal{C})_{/T} \simeq V_T^{-1}\mathcal{C}_{/T}.$$

To say that the localization $\mathcal{C} \rightarrow V^{-1}\mathcal{C}$ is locally cartesian closed means that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}_{/S} & \xrightarrow{L_S} & V_S^{-1}\mathcal{C}_{/S} \\ f^* \downarrow & & \downarrow f^* \\ \mathcal{C}_{/T} & \xrightarrow{L_T} & V_T^{-1}\mathcal{C}_{/T}. \end{array}$$

³³[HA 1.3.4.22]

³⁴[HA 1.3.4.26]

Hence

$$L_T f^* : \mathcal{C}_{/S} \rightarrow V_T^{-1} \mathcal{C}_{/T}$$

inverts maps in V_S (i.e. carries every morphism in V_T to an equivalence). Since a left Bousfield localization along a strongly saturated class V_T inverts precisely the maps in V_T , and the passage from \mathcal{M} to $N(\mathcal{M})[W^{-1}]$ inverts precisely the maps in W , it follows that, on the level of model categories,

$$f^* : \mathcal{M}_{/S} \longrightarrow \mathcal{M}_{/T}$$

sends V_S -local equivalences to V_T -local equivalences, as required. \square

6.6 Constructing models. Given a presentable ∞ -category \mathcal{C} , one can use its “standard presentation” as a localization of a presheaf ∞ -category to construct a combinatorial simplicial model category \mathcal{M} whose underlying ∞ -category is equivalent to \mathcal{C} . This model category can be obtained as a left Bousfield localization of a simplicial presheaf (model) category, essentially by copying the “generators and relations”. While this construction is not unique, it is sort of standard, and we shall refer only to this construction when talking about a model of \mathcal{C} . We recall the construction³⁵ since we need some details of it. Since \mathcal{C} is presentable, it is by definition an accessible localization (for some sufficiently large cardinal κ) of the ∞ -category $\text{Pre}(\mathcal{C}^\kappa)$ of presheaves (of ∞ -groupoids) on the ∞ -category \mathcal{C}^κ of κ -compact objects in \mathcal{C} . More specifically, writing U for the (small) set of maps of the form

$$\text{colim}_i \text{Map}(-, x_i) \longrightarrow \text{Map}(-, \text{colim}_i x_i),$$

for $x : I \rightarrow \mathcal{C}^\kappa$ a diagram indexed on a κ -small simplicial set I , we have an equivalence $\mathcal{C} \simeq U^{-1} \text{Pre}(\mathcal{C}^\kappa)$. Note that $\text{Pre}(\mathcal{C}^\kappa) \rightarrow \mathcal{C}$ is locally cartesian by Proposition 1.5.

Let \mathfrak{C} be a fibrant simplicial category such that $\mathcal{C}^\kappa \simeq N(\mathfrak{C})$, and consider the simplicial category $\text{Pre}_\Delta(\mathfrak{C})$ of simplicial presheaves on \mathfrak{C} , endowed with the injective model structure. Let V be the smallest saturated class generated by the set of maps of the form

$$\text{hocolim}_i \text{Map}(-, x_i) \longrightarrow \text{Map}(-, \text{hocolim}_i x_i)$$

in $\text{Pre}_\Delta(\mathfrak{C})$ corresponding to the maps in U , and let \mathcal{M} be the left Bousfield localization of $\text{Pre}_\Delta(\mathfrak{C})$ by V . This \mathcal{M} is a model for \mathcal{C} , in the sense that we have equivalences

$$\mathcal{C} \simeq U^{-1} \text{Pre}(\mathcal{C}^\kappa) \simeq V^{-1} N(\text{Pre}_\Delta(\mathfrak{C})) [W^{-1}],$$

where W denotes the weak equivalences in $\text{Pre}_\Delta(\mathfrak{C})$. It is also clear that the induced localization on the ∞ -level is precisely $\text{Pre}(\mathcal{C}^\kappa) \rightarrow \mathcal{C}$.

Our main theorem in this section is the following.

Theorem 6.7 *Let \mathcal{C} be a presentable locally cartesian closed ∞ -category. Let \mathcal{M} be the combinatorial simplicial model category constructed above, whose underlying ∞ -category is equivalent to \mathcal{C} . Then \mathcal{M} is a type-theoretic model category.*

Proof. We shall apply Proposition 6.5 to the left Bousfield localization $\text{Pre}_\Delta(\mathfrak{C}) \rightarrow \mathcal{M}$, so we must first establish that $\text{Pre}_\Delta(\mathfrak{C})$ is itself a type-theoretic model category. First of all, $\text{Pre}_\Delta(\mathfrak{C})$ is locally cartesian closed as an ordinary category: the simplicial category \mathfrak{C} can be viewed as an internal category object in Set_Δ , and accordingly $\text{Pre}_\Delta(\mathfrak{C})$ can be viewed as the category of presheaves on \mathfrak{C} internally to Set_Δ . But this is always a topos.³⁶ Second, $\text{Pre}_\Delta(\mathfrak{C})$ is proper and has the monos as cofibrations since the model category of simplicial sets (with its standard “Kan” model structure) satisfies these conditions. So $\text{Pre}_\Delta(\mathfrak{C})$ is a type-theoretic model category. Now the left Bousfield localization $\text{Pre}_\Delta(\mathfrak{C}) \rightarrow \mathcal{M}$ induces the standard presentation $\text{Pre}(\mathcal{C}^\kappa) \rightarrow \mathcal{C}$ at the ∞ -category level, and since this localization is locally cartesian by Proposition 1.5, we conclude by Proposition 6.5 that \mathcal{M} is a type-theoretic model category. \square

³⁵[HTT, Prop. A.3.7.6]

³⁶See for example Theorem 6.5 of Barr–Wells [4].

6.8 Univalence in type-theoretic model categories. Given a univalent family $p : X \rightarrow S$ in a presentable locally cartesian closed ∞ -category \mathcal{C} , we show how to lift it to the model. Taking the viewpoint that the intrinsic content is at the level of the underlying ∞ -category, we could simply *define* univalence in a type-theoretic model category \mathcal{M} by saying that a fibration $p : X \rightarrow S$ between fibrant objects in \mathcal{M} is called *univalent* if it is univalent as a map in the underlying ∞ -category $N(\mathcal{M})[W^{-1}]$. From this viewpoint it is easy to see that starting from any univalent family $p : X \rightarrow S$ in ∞ -category \mathcal{C} , any fibrant representative in \mathcal{M} is a univalent fibration.

However, from the viewpoint of type theory, univalence is about identity types, and it translates into a condition on type-theoretic model categories which does not explicitly refer to the underlying ∞ -category. We therefore provide another formulation of univalence, in a style more directly related to the notion as it appears for example in [13] and [16].

We will write \mathcal{H} for the homotopy category of simplicial sets, which we view as a cartesian closed category. In particular, \mathcal{H} is a symmetric monoidal category under the cartesian product, and is canonically enriched over itself via the exponential. If \mathcal{C} is a given \mathcal{H} -enriched category, we may consider the category $\text{Pre}_{\mathcal{H}}(\mathcal{C})$ of \mathcal{H} -enriched presheaves on \mathcal{C} ; if \mathcal{C} is large, then $\text{Pre}_{\mathcal{H}}(\mathcal{C})$ is very large, but we will not need to actually work with $\text{Pre}_{\mathcal{H}}(\mathcal{C})$. Note that \mathcal{H} admits isomorphism objects; that is, given a pair of objects X and Y and \mathcal{H} , there exists an object $\text{Eq}(X, Y)$ of homotopy equivalences between X and Y , defined as the subobject of

$$\text{Map}(X, Y) \times \text{Map}(Y, X)$$

consisting of the inverse equivalences. (This is the symmetric definition, but by uniqueness of inverses it may be equivalently defined as a subobject of either $\text{Map}(X, Y)$ or $\text{Map}(Y, X)$ consisting of the equivalences.) The same construction gives an object $\text{Eq}(X, Y)$ for any pair of homotopy classes of objects of a model category \mathcal{M} .

Now, given a type-theoretic model category \mathcal{M} and a fibration $p : X \rightarrow S$ between fibrant objects of \mathcal{M} , consider the category $\text{Pre}_{\mathcal{H}}(\text{Ho}(\mathcal{M}_{/S \times S}))$ of \mathcal{H} -enriched presheaves on $\text{Ho}(\mathcal{M}_{/S \times S})$. The diagonal $S \rightarrow S \times S$, as an object of $\text{Ho}(\mathcal{M}_{/S \times S})$, represents an \mathcal{H} -enriched presheaf on $\text{Ho}(\mathcal{M}_{/S \times S})$, and we also have the \mathcal{H} -enriched presheaf which sends the object $(f, g) : T \rightarrow S \times S$ of $\text{Ho}(\mathcal{M}_{/S \times S})$ to the homotopy type

$$\text{Eq}_{/T}(f^*X, g^*X)$$

of the space of equivalences over T between f^*X and g^*X . Writing $\text{Map}_{/S \times S}(-, S)$ for the former and $\underline{\text{Eq}}_{/S}(X, X)$ for the latter, it follows from the \mathcal{H} -enriched Yoneda lemma that the identity equivalence $\text{id}_X : X \rightarrow X$ over S induces a map of presheaves

$$u : \text{Map}_{/S \times S}(-, S) \longrightarrow \underline{\text{Eq}}_{/S}(X, X).$$

This leads us to the following rigidification of the notion of univalence, adapted to the setting of type-theoretic model categories.

Definition 6.9 Let \mathcal{M} be a type-theoretic model category and let $p : X \rightarrow S$ be a fibration between fibrant objects of \mathcal{M} . Then p is said to be *univalent* if the map

$$u : \text{Map}_{/S \times S}(-, S) \longrightarrow \underline{\text{Eq}}_{/S}(X, X).$$

is an isomorphism of \mathcal{H} -enriched presheaves.

Proposition 6.10 *Let \mathcal{M} be a type-theoretic model category. A fibration $p : X \rightarrow S$ between fibrant objects of \mathcal{M} is univalent in the model category \mathcal{M} if and only if it is univalent in the underlying ∞ -category $N(\mathcal{M})[W^{-1}]$.*

Proof. In general, if $\mathcal{C} \rightarrow \mathcal{D}$ is a functor of ∞ -categories, then it induces a functor $\text{Ho}(\mathcal{C}) \rightarrow \text{Ho}(\mathcal{D})$ of \mathcal{H} -enriched categories. Moreover, this assignment is natural, in that it induces a functor

$$\text{Ho}(\text{Fun}(\mathcal{C}, \mathcal{D})) \longrightarrow \text{Fun}_{\mathcal{H}}(\text{Ho}(\mathcal{C}), \text{Ho}(\mathcal{D})),$$

where the latter denotes the category of \mathcal{H} -enriched functors which is conservative, since a (homotopy class of) natural transformation of functors from \mathcal{C} to \mathcal{D} is a natural equivalence if and only if the resulting natural transformation of \mathcal{H} -enriched functors $\mathrm{Ho}(\mathcal{C}) \rightarrow \mathrm{Ho}(\mathcal{D})$ is a natural isomorphism.

In particular, taking $\mathcal{C} = \mathcal{N}(\mathcal{M}_{/S \times S})[W^{-1}]^{\mathrm{op}}$ and $\mathcal{D} = \mathcal{N}(\mathrm{Set}_{\Delta})[W^{-1}]$, we obtain a conservative functor

$$\mathrm{Ho}(\mathrm{Pre}(\mathcal{N}(\mathcal{M}_{/S \times S})[W^{-1}])) \longrightarrow \mathrm{Pre}_{\mathcal{H}}(\mathrm{Ho}(\mathcal{M}_{/S \times S})).$$

By definition, $p : X \rightarrow S$ is univalent in \mathcal{M} if and only if u is a natural isomorphism in $\mathrm{Pre}_{\mathcal{H}}(\mathrm{Ho}(\mathcal{M}_{/S \times S}))$, if and only if u is a natural equivalence in $\mathrm{Ho}(\mathrm{Pre}(\mathcal{N}(\mathcal{M}_{/S \times S})[W^{-1}]))$. \square

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